Group analysis for generalized reaction-diffusion convection equation

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Abstract

In this paper we discuss about group classification for non-linear generalized reaction-diffusion convection equation: $u_t = (f(x,u)u_x)_x + h(x,u)u_x + k(x,u)$, by using Lie-classical symmetry method. For this, first we find its symmetry group and then we find differential invariants for resulted group by using infinitesimal criterion method and at the end reduce modeling equations by using resulted invariants. We present application of this group classification in group classification and obtaining related similarity solution of KPP equation, too.

Key words: symmetry, group classification, differential invariants, Lie-classical method,infinitesimal criterion method, RDC equation, KPP equation, similarity solutions.

1 Introduction

This paper devoted to group classification of Generalized Reaction-Diffusion Convection (G-RDC) equation, by using Lie-classical method.

$$\Delta\ :\ u_t = (f(x,u)u_x)_x + h(x,u)u_x + k(x,u), \tag{1.1}$$

Where $u(x,t)$ is unknown function and $f(x,u), h(x,u), k(x,u)$ are arbitrary functions. The equation (1.1) generalizes a number of the well known second-order evolution equations, describing various process in physics, chemistry and biology. Symmetry group method plays an important role in the analysis of differential equations. The history of group classification methods goes back to Sophus Lie [9]. (See [2,4,7,6]). His work devoted to finding symmetry groups, differential invariants and linearized or reduced equation for given model. There are several approach to group classification of differential equation, we apply infinitesimal method (See [2,4,10]) for this. There are another useful articles and accounts about group classification for similar equations of (1.1) via other methods, (See [11,12,13]). In this paper we generalize RDC equation to G-RDC equation and applay Lie-classical symmetry method via applied approach. we hope this work be useful to applied and theoretical readers.

2 Group classification for modeling equation

Let following one-parameter group

$$\overline{x} = x + \varepsilon\xi(x,t,u) + O(\varepsilon^2), \qquad \overline{t} = t + \varepsilon\eta(x,t,u) + O(\varepsilon^2), \qquad \overline{u} = u + \varepsilon\varphi(x,t,u) + O(\varepsilon^2), \tag{2.2}$$

be symmetry group of modeling equation Δ. We can obtain ξ, η and φ, by using infinitesimal method.

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Consider the vector field $X := \xi\partial_x + \eta\partial_t + \varphi\partial_u$ in total space $M = (x, t, u)$ with $p = 2$ and $q = 1$. If this vector field be an infinitesimal generator of Δ's symmetry group, then

$$X^{(2)}\Delta = 0, \qquad \text{whenever} \qquad \Delta = 0. \tag{2.3}$$

Where $X^{(2)}$ is second prolong of X and has following form:

$$X^{(2)} = X + \varphi^x\partial_{u_x} + \varphi^t\partial_{u_t} + \varphi^{xx}\partial_{xx} + \varphi^{xt}\partial_{u_{xt}} + \varphi^{tt}\partial_{u_{tt}}, \tag{2.4}$$

Where $\varphi^x, \varphi^t, \varphi^{xx}, \varphi^{xt}$ and φ^{tt} are respectively:

$$\begin{aligned}
&\varphi^x = D_xQ + \xi u_{xx} + \eta u_{xt}, \qquad \varphi^t = D_tQ + \xi u_{xt} + \eta u_{tt},\\
&\varphi^{xx} = D_{xx}Q + \xi u_{xxx} + \eta u_{xxt}, \qquad \varphi^{xt} = D_{xt}Q + \xi u_{xxt} + \eta u_{xtt}, \qquad \varphi^{tt} = D_{tt}Q + \xi u_{xtt} + \eta u_{ttt}.
\end{aligned}$$

Where D_x, D_t are total derivative with respect to specified variables, $D_{xx} = D_xD_x$, $D_{xt} = D_xD_t$ and $D_{tt} = D_tD_t$, and $Q = \varphi - \xi u_x - \eta u_t$ is the corresponding characteristic of X (See [2,4,7,6]). By using criterion (2.4), we find:

$$\begin{aligned}
\varphi^t = {} & (f_{xx}u_x + f_{xu}u_x^2 + f_xu_{xx} + k_x + h_xu_x)\xi + (f_{xu}u_x + f_{uu}u_x^2 + f_uu_{xx} + h_uu_x + k_u)\varphi + \\
& +(f_x + 2f_uu_x + h)\varphi^x + f\varphi^{xx}.
\end{aligned} \tag{2.5}$$

By substituting $\varphi^x, \varphi^t, \varphi^{xx}, \varphi^{xt}$ and φ^{tt} in (2.5), we have following results:

coefficient	monomial
1	$\varphi_t - f_x\varphi_x - f\varphi_{xx} - h\varphi_x - k_x\xi - k_u\varphi$
u_x	$\xi_t + f_{xx}\xi + f_{xu}\varphi + f_x(\varphi_u - \xi_x) + 2f_u\varphi_x + f(2\varphi_{xu} - \xi_{xx}) + h(\varphi_u - \xi_x) + h_x\xi + h_u\varphi$
u_t	$\varphi_u - \eta_t + f_x\eta_x + f\eta_{xx} + h\eta_x$
u_xu_t	$\xi_u - f_x\eta_u - 2f_u\eta_x - f\eta_{xu} - h\eta_u$
u_t^2	η_u
u_x^2	$-f_x\xi_u + f_{xu}\xi + f_{uu}\varphi + 2f_u(\varphi_u - \xi_x) - h\xi_u + f(\varphi_{uu} - 2\xi_{xu})$
u_x^3	$2f_u\xi_u + f\xi_{uu}$
$u_x^2u_t$	$2f_u\eta_u - f\eta_{uu}$
u_{xx}	$f_x\xi + f_u\varphi + f(\varphi_u - 2\xi_x)$
u_xu_{xt}	$2f\eta_x$
u_xu_{xx}	$3f\xi_u$
u_tu_{xx}	$f\eta_u$
$u_{xx}u_x^2$	$2f\eta_u$

(Table 1)

By simplifying above equations we obtain:

$$\begin{aligned}
&\eta = \eta(t), \qquad \xi = \xi(x, t), \qquad \varphi_u - \eta_t = 0, \qquad f_x\xi + f_u\varphi + f(\varphi_u - 2\xi_x) = 0,\\
&\varphi_t - f_x\varphi_x - f\varphi_{xx} - h\varphi_x - k_x\xi - k_u\varphi = 0, \qquad f_{xu}\xi + f_{uu}\varphi + 2f_u(\varphi_u - \xi_x) = 0,\\
&\xi_t + f_{xx}\xi + f_{xu}\varphi + (f_x + h)(\varphi_u - \xi_x) + 2f_u\varphi_x - f\xi_{xx} + h_x\xi + h_u\varphi = 0.
\end{aligned} \tag{2.6}$$

3 Group classification in special cases

In this section we consider four special case of modeling equation and obtain differential invariants for them by using (2.6) and infinitesimal criterion method.

A: $f(x, u) = xu^{-1}$, $h(x, u) = -2/u$, $k(x, u) = au + b$, where a, b are constant real numbers. In this case we have: $\eta = \frac{1}{a}e^{at}.e^{ac_1} + c_2$, $\xi = c_3\sqrt{x}$, and $\varphi = u.e^{at}.e^{ac_1}$. As a result we find 3 independent vector fields: $X_1 = e^{at}\partial_t + ae^{at}\partial_u$, $X_2 = \partial_t$, $X_3 = \sqrt{x}\partial_x$.

B: $f(x,u) = ax^4u$, $h(x,u) = \frac{bx}{u}$, $k(x,u) = xu$, where $a \neq 0, b$ are real numbers. In this case we have: $\eta = -c_1t + c_2$, $\xi = c_1x$, $\varphi = -c_1u$. As a result we find 2 independent vector fields: $X_1 = -t\partial_t + x\partial_x - u\partial_u$, $X_2 = \partial_t$.

C: $f(x,u) = ax\exp(-u/b)$, $h(x,u) = xu$, $k(x,u) = c - bu$, where $a \neq, b \neq 0, c$ are real constant numbers. In this case we have: $\eta = c_1$, $\xi = -\frac{c_2}{c}x\exp(bt)$, $\varphi = c_2\exp(bt)$. As a result we find 2 independent vector fields: $X_1 = -\frac{1}{b}x\exp(bt)\partial_x + \exp(bt)\partial_u$, $X_2 = \partial_t$.

D: $f(x,u) = ax^2u$, $h(x,u) = xu$, $k(x,u) = u$, where a is real nonzero constant. In this case we have: $\eta = c_1$, $\xi = c_2x$, $\varphi = 0$. As a result we find 2 independent vector fields: $X_1 = \partial_t$, $X_2 = x\partial_x$.

Similar to above, the reader can use above procedure for finding her or him interested modeling equation where has form (1.1), with interested f, h and k.

4 Resulted differential invariants

In this section we obtain differential invariants for above resulted symmetry groups in several major and complicated cases. For example we compute differential equation for **B**, X_1 and **C**, X_1.

B, X_1: In this case we have following determination equation: $\frac{dx}{x} = \frac{dt}{-t} = \frac{du}{-u}$, and by solving this equation we find: $xt = c_1$, $xu = c_2$, $u/t = c_3$; and we choose $r = xt$ and $w = xu$ as independent invariants. (we note $u/x = w/r$ and as a result obtain from r, w.)

C, X_1: In this case we have following determination equation: $\frac{bdx}{x\exp(bt)} = \frac{dt}{0} = \frac{du}{\exp(bt)}$, and by solving this equation we find : $t = c_1$, $c_2 = u + b\ln x$; and we choose $r = t$ and $w = u + b\ln x$ as independent invariants.

case	interested vector field	differential invariants
A	X_1	$r = x$, $w = u - at$
	X_2	$r = x$, $w = u$
	X_3	$r = t$, $w = u$
B	X_1	$r = xt$, $w = xu$
	X_2	$r = x$, $w = u$
C	X_1	$r = t$, $w = u + b\ln x$
	X_2	$r = x$, $w = u$
D	X_1	$r = x$, $w = u$
	X_2	$r = t$, $w = u$

(Table 2)

In the above table, F is an arbitrary function.

In the sequel, we obtain reduced equation respect to specified group symmetry with infinitesimal generator X, (solution of this reduced equation called X-invariants solution of original equation) by using resulted differential invariants in the above table in two case.

For example, consider $u_t = (ax^4u)_x + bx/uu_x + xu$ (case B). By considering $w = w(r)$, we find: $u_t = w_r$, $u_x = (xtw_r - w)/x^2$ and $u_{xx} = (x^2(tw_r + xt^2w_{rr} - tw_r) - 2x(xtw_r - w))/x^4$. By substituting this values in the given equation, we find following X_1-reduced equation:

$$w_r = b + (1-4a)w + (6a + ar^3)w^2 + (4ar - 2aw + awr - 2arw)w_r + awr(r-a)w_{rr}$$

As and second example, consider $u_t = \left(\frac{ax}{\exp(u/b)}\right)_x + xuu_x + c - bu$, By considering $w = w(r)$, we find: $u_t = w_r$, $u_x = -1/x$ and $u_{xx} = 1/x^2$. By substituting this values in the given equation, we find following X_1-reduced equation:

$$w_r = c + ab - 2a,$$

5 Some Applications

The Kolomogorov-Petrovskii-Piskonov (KPP) equation, (See [1,8])

$$E(u) \equiv bu_t - u_{xx} + \gamma uu_x + f(u), \tag{5.7}$$

with (b,γ) real numbers, is encountered in reaction-diffusion systems and prey-predator models. The optional convection term uu_x [1,4]) is quite important in physical applications to prey-predator models.

5.1 Classical symmetries and Differential invaiants

If we let $b \neq 0$, then we have following equation:

$$u_t = \frac{1}{b}(u_{xx} - \gamma uu_x - f(u)), \tag{5.8}$$

By substituting this value in (2.6), we have following results.

Case I: $b = \frac{\alpha\gamma}{\exp(\beta\alpha)}, f(u) = \frac{(1/2)\gamma\kappa\alpha u}{\exp(\alpha\beta)} + s$; Where α, β, κ and s are arbitrary constants.

In this case we have:

$$\xi = \frac{\exp(\alpha t)\exp(\alpha\beta)}{\alpha} + c_2, \qquad \eta = c_1, \qquad \varphi = \kappa\exp(\alpha t), \tag{5.9}$$

For symmetry algebra we find:

$$X_1 = \partial_t \qquad X_2 = \partial_x, \tag{5.10}$$

Case II: $b \neq \frac{\alpha\gamma}{\exp(\beta\alpha)}, f(u) \neq \frac{(1/2)\gamma\kappa\alpha u}{\exp(\alpha\beta)} + s$; Where α, β, κ and s are arbitrary constants.

In this case we have:

$$\xi = c_1 \qquad \eta = c_2, \qquad \varphi = 0, \tag{5.11}$$

For symmetry algebra we find:

$$X_1 = \partial_t \qquad X_2 = \partial_x, \tag{5.12}$$

As a result we have following theorem:

Theorem 1 *Some exact solutions for modeling equation (5.7) invariant under a translation group respect to x and some solutions of this equation invariant under translation respect to t.*

5.2 Similarity solutions

In this subsection we find similarity solution of equation (5.8) by using above resulted symmetry algebra.

similarity solution respect to $X = \partial_t$. In this case we have following equation as X-reduced equation:

$$w_{rr} - \gamma w w_r - f(w) = 0, \tag{5.13}$$

If we solve equation (5.13) with MAPLE, then we find: $w(x) = c$. Where

$$\frac{d}{dc}F(c)F(c) - \gamma(cF(c)) - f(c) = 0 \quad \text{or} \quad \frac{d}{dr}w(r) = F(c), \quad \text{or} \quad r = \int \frac{1}{F(c)}dc + C \tag{5.14}$$

Where F is arbitrary function with specified arguments and c, C are arbitrary constants.

similarity solution respect to $Y = \partial_x$. In this case we have following equation as Y-reduced equation:

$$w_r + \frac{1}{b}f(w) = 0, \tag{5.15}$$

If we solve equation (5.15) with MAPLE, then we find following solution:

$$x - \int^{w(x)} \frac{b}{f(c_1)}dc_1 + c_2 = 0, \tag{5.16}$$

Where c_1 and c_2 are arbitrary constants.

Conclusion

In this paper first we find system of equations to finding symmetry group and symmetry algebra for (G-RDC) equation, then obtain these symmetry groups in several special cases and at the end we establish symmetry classification for KPP equation by using group classification of (G-RDC) equation and we find its similarity solution respect to resulted symmetry algebra.

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